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A search for shape-invariant solvable potentials

G Lévai†

Department of Theoretical Physics, University of Oxford, Oxford OX1 3NP, UK

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Abstract. We investigate a simple method of constructing potentials which is related to the work of Bhattacharjie and Sudarshan and for which the Schrödinger equation can be solved in terms of known special functions. It turns out that this method can be related to supersymmetric quantum mechanics and this relationship can help us to decide which special functions, satisfying linear homogeneous second-order differential equations, can be solutions of the Schrödinger equation with potentials of the form $V(x) = W^2(x) - W'(x)$. We illustrate this procedure with the example of orthogonal polynomials and obtain explicit expressions of wavefunctions of a wide class of shape-invariant potentials.

1. Introduction

Recently there has been renewed interest in simple quantum mechanical systems as a result of the introduction of two important concepts: supersymmetric quantum mechanics (see, for example, Witten 1981, Cooper and Freedman 1983, Andrianov *et al* 1984) and shape invariance (Gendenshtein 1983). In the formalism of supersymmetric quantum mechanics, Hamiltonians of two systems are connected by supersymmetry transformations. Due to this symmetry the spectra of the two Hamiltonians are identical, except for the ground state. A significant development was the introduction of the concept of shape invariance. If the Hamiltonians related by supersymmetry satisfy the shape-invariance condition, the spectra and the wavefunctions can be determined by elementary calculations. It turned out that most of the known solvable potentials are shape invariant. Nevertheless shape invariance is not a general feature of solvable potentials (Cooper *et al* 1987). A systematic study of the relationship between solvability and shape invariance has been carried out by Cooper *et al* (1987) for the Natanzon class potentials (whose solutions are hypergeometric functions).

One can ask whether there are any other special functions which are solutions of the Schrödinger equation with shape-invariant potentials. Here we try to answer this question by investigating a simple method of finding solvable potentials. With the help of this method the Schrödinger equation can be transformed into various linear homogeneous second-order differential equations. First we try to link this method with the formalism of supersymmetric quantum mechanics and deduce a condition which has to be satisfied by any special function in order to give potentials of the form $V(x) = W^2(x) \pm W'(x)$, the standard expression of potentials in supersymmetric quantum mechanics. Then we investigate whether these potentials satisfy the shape-invariance condition. These procedures are illustrated with the example of orthogonal polynomials as special functions.

† Permanent address: Institute of Nuclear Research of the Hungarian Academy of Sciences, Debrecen, POB 51, Hungary 4001.

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The arrangement of this paper is as follows. In § 2 we give a brief survey of supersymmetric quantum mechanics and collect the formulae necessary for subsequent sections. In § 3 we present a simple method of obtaining solvable potentials (Bhattacharjie and Sudarsan 1962) and link it with the formalism of supersymmetric quantum mechanics. We apply this method to the Jacobi, generalised Laguerre and Hermite polynomials in § 4 and find a potential which escaped notice in other publications. We also show that the potentials obtained here are examples of the different factorisation types of Infeld and Hull (1951) and compare different classification schemes of the same families of potentials (Miller 1968, Cooper *et al* 1987). We turn to the question of shape invariance in § 5. Finally we summarise the results in § 6.

2. A brief survey of supersymmetric quantum mechanics

In supersymmetric quantum mechanics two Hamiltonians related by supersymmetry have the form ($\hbar = 2m = 1$)

$$H_{\pm} = -\frac{d^2}{dx^2} + V_{\pm}(x) \tag{2.1}$$

where

$$V_{\pm}(x) = W^2(x) \pm W'(x). \tag{2.2}$$

The partner Hamiltonians can be factorised as

$$H_- = A^\dagger A \qquad H_+ = AA^\dagger \tag{2.3}$$

where

$$A = \frac{d}{dx} + W(x)$$

$$A^\dagger = -\frac{d}{dx} + W(x). \tag{2.4}$$

In the case of unbroken supersymmetry the ground state of H_- has zero energy ($E_0^{(-)} = 0$) and the ground-state wavefunction is related to the superpotential $W(x)$ as

$$W(x) = -(\ln \Psi_0^{(-)})'. \tag{2.5}$$

The energy eigenvalues of H_- and H_+ are identical, except for the ground state:

$$E_n^{(+)} = E_{n+1}^{(-)} \qquad n = 0, 1, \ldots; E_0^{(-)} = 0. \tag{2.6}$$

The eigenfunctions of $H_-(\Psi_n^{(-)})$ and $H_+(\Psi_n^{(+)})$ are connected by operators $A^\dagger$ and A:

$$\begin{aligned} A\Psi_{n+1}^{(-)}(x) &= (E_n^{+})^{1/2}\Psi_n^{(+)}(x) \\ A^\dagger\Psi_n^{(+)}(x) &= (E_n^{+})^{1/2}\Psi_{n+1}^{(-)}(x). \end{aligned} \tag{2.7}$$

Now let us turn to the question of shape invariance. Potentials are called shape invariant if their dependence on x, the coordinate, is similar and they differ only in some parameters which appear in them. In particular $V_+(x, a)$ and $V_-(x, a)$ are shape invariant if they satisfy the shape-invariance relationship:

$$V_+(x, a_0) - V_-(x, a_1) = W^2(x, a_0) + W'(x, a_0) - W^2(x, a_1) + W'(x, a_1) = R(a_1) \tag{2.8}$$

where the sets of parameters a_1 and a_0 are related by a functional:

$$a_1 = f(a_0). \tag{2.9}$$

It is easy to show (Gendenshtein 1983) that the energy eigenvalues of H_- are given by

$$\begin{aligned} E_n^{(-)} &= \sum_{k=1}^{n} R(a_k) \\ a_k &= f^k(a_0). \end{aligned} \tag{2.10}$$

The eigenfunctions can also be expressed in a simple way (Dutt *et al* 1986):

$$\Psi_n^-(x, a_0) = N_0 A^\dagger(x, a_0) A^\dagger(x, a_1) \ldots A^\dagger(x, a_{n-1}) \Psi_0^{(-)}(x, a_n). \tag{2.11}$$

Dabrowska *et al* (1988) found the explicit expression of the wavefunctions of a wide class of shape-invariant potentials using these operator techniques.

Before closing this section we mention some recent applications of supersymmetric quantum mechanics. Several authors studied the Coulomb and harmonic oscillator problems (D'Hoker and Vinet 1985, Lahiri *et al* 1987, Ding 1987, Engelfield 1988, Amado 1988) and the relationship of these systems in arbitrary dimensions (Kostelecký *et al* 1985) from the point of view of supersymmetric quantum mechanics. These systems are especially interesting, since they have a well established group structure (see, for example, Wybourne 1974) and it is a straightforward task to investigate the relationship between the well known Lie algebras related to them and the graded Lie algebras (see, for example, Kac 1977) which give the mathematical framework of supersymmetry.

There are applications making use of supersymmetric quantum mechanics to describe scattering states and scattering amplitudes (Sukumar 1985a, b, 1986, 1987, Khare and Sukhatme 1988).

3. A simple method of obtaining solvable potentials

Consider the Schrödinger equation ($\hbar = 2m = 1$)

$$\frac{d^2\Psi}{dx^2} + (E - V(x))\Psi = 0. \tag{3.1}$$

Its solutions generally take the form

$$\Psi(x) = f(x) F(g(x)) \tag{3.2}$$

where $F(g)$ is a special function which satisfies a second-order differential equation:

$$\frac{d^2F}{dg^2} + Q(g)\frac{dF}{dg} + R(g)F(g) = 0. \tag{3.3}$$

The form of $Q(g)$ and $R(g)$ is well defined for any special function $F(g)$. Substituting $\Psi(x) = f(x)F(g(x))$ into (3.1) leads to the second-order differential equation

$$\frac{d^2F}{dg^2} + \frac{dF}{dg}\left(\frac{g''}{(g')^2} + \frac{2f'}{fg'}\right) + F(g)\left(\frac{f''}{f(g')^2} + \frac{E - V(x)}{(g')^2}\right) = 0. \tag{3.4}$$

But from (3.3) it follows that

$$\frac{g''}{(g')^2} + \frac{2f'}{g'f} = Q(g(x)) \tag{3.5}$$

and

$$\frac{f''}{f(g')^2}+\frac{E-V(x)}{(g')^2}=R(g(x)) \tag{3.6}$$

so we get the following formula:

$$E-V(x)=R(g(x))(g')^2-f''/f' \tag{3.7}$$

$$=R(g(x))(g')^2-((f'/f)^2+(f'/f)'). \tag{3.8}$$

Since f'/f can be expressed explicitly from (3.5) we can obtain $E-V(x)$ in terms of $g(x)$, $Q(g(x))$ and $R(g(x))$:

$$E-V(x)=\frac{g'''}{2g'}-\frac{3}{4}\left(\frac{g''}{g'}\right)^2+(g')^2\left(R(g(x))-\frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}g}-\tfrac{1}{4}Q^2(g(x))\right). \tag{3.9}$$

This means that, once choosing $Q(g)$ and $R(g)$ (e.g. the type of the special function $F(g)$), we can experiment with different internal functions $g(x)$ to see whether we can get reasonable potentials. In the next section some examples of finding appropriate $g(x)$ will be given. From (3.5) we can readily calculate $f(x)$ as well:

$$f(x)\simeq(g')^{-1/2}\exp\left(\frac{1}{2}\int^{g(x)}Q(g)\,\mathrm{d}g\right). \tag{3.10}$$

This simple method of investigating the solution of the Schrödinger equation has been known for a long time (Bhattacharjie and Sudarshan 1962). These authors applied this method to the hypergeometric, confluent hypergeometric and Bessel equations. Later it turned out that it can be related to algebraic methods of solving differential equations (Cordero *et al* 1971). Another systematic application of this method (to the hypergeometric functions) has been carried out by Natanzon (1971, 1979) independently.

From (3.8) it is clear that this method is closely related to the theory of supersymmetric quantum mechanics, since in any case when $R(g(x))=0$ holds, we get

$$E-V(x)=-W^2(x)+W'(x) \tag{3.11}$$

where

$$W(x)=-(\ln(f))'. \tag{3.12}$$

Thus, investigating the structure of $R(g(x))$ can help us to decide which special functions can lead to potentials of the form $W^2(x)-W'(x)$. Orthogonal polynomials seem to be especially suited to this problem, since $R(n,g(x))=n\gamma(n,g(x))$ holds for them. Thus, for $n=0$, $R(x)$ vanishes. In this case $E_{n=0}=0$ and $f(x)=\Psi_0(x)$. This means that one can generally factor out $\Psi_0(x)$, the ground-state wavefunction, from $\Psi(x)$, a result obtained by means of operator techniques by Dabrowska *et al* (1988). In cases when $R(n,g(x))(g')^2=\text{constant}$, $f(x)=\Psi_0(x)$ for any n. When $R(n,g(x))(g')^2$ is not constant, then $f(x)=\Psi_0(x)\phi(x)$ and $R(n,g(x))(g')^2$ contributes to E_n and at the same time it kills off the extra coordinate dependence which arises from the presence of $\phi(x)$ on the right-hand side of (3.8).

Similar considerations can be applied to other special functions satisfying second-order differential equations. We shall discuss this in § 4.

4. Systematic study of potential problems with solutions containing orthogonal polynomials

In our treatment we take only the Jacobi, generalised Laguerre and Hermite polynomials ($P_n^{(\alpha,\beta)}(g)$, $L_n^{(\alpha)}(g)$ and $H_n(g)$) into account, since the other orthogonal polynomials (Gegenbauer, Chebyshev, Legendre) can be obtained as special cases from $P_n^{(\alpha,\beta)}(g)$.

We apply the procedure described in § 3 first to the Jacobi polynomials. It will turn out that we can get all the shape-invariant potentials obtained by Dabrowska *et al* (1988) and one more. There is an intimate relationship between the Jacobi polynomials and the hypergeometric function (Abramowitz and Stegun 1970). Any wavefunction expressed in terms of Jacobi polynomials can also be expressed in terms of hypergeometric functions as well. Nevertheless, in some cases it is more convenient to use these polynomials, because a wide class of solvable potentials can be found more easily if we take them as a starting point. At the same time, considerations presented at the end of the preceding section showed that the aspects of supersymmetric quantum mechanics are more transparent in the case of orthogonal polynomials.

From the differential equation of the Jacobi polynomials (Abramowitz and Stegun 1970) one can see that

$$Q(g)=\frac{\beta-\alpha}{1-g^2}-(\alpha+\beta+2)\frac{g}{1-g^2} \tag{4.1}$$

and

$$R(g)=\frac{n(n+\alpha+\beta+1)}{1-g^2}. \tag{4.2}$$

Substituting these into (3.9) yields

$$\begin{aligned} E-V(x)=&\frac{1}{2}\frac{g'''}{g'}-\frac{3}{4}\left(\frac{g''}{g'}\right)^2+\frac{(g')^2}{(1-g^2)}n(n+\alpha+\beta+1)\\ &+\frac{(g')}{(1-g^2)^2}[\tfrac{1}{2}(\alpha+\beta+2)-\tfrac{1}{4}(\beta-\alpha)^2]\\ &+\frac{(g')^2g}{(1-g^2)^2}[\tfrac{1}{2}(\beta-\alpha)(\beta+\alpha)]+\frac{(g')^2g^2}{(1-g^2)^2}\left(\frac{1}{4}-\left(\frac{\alpha+\beta+1}{2}\right)^2\right). \end{aligned} \tag{4.3}$$

Since we have to get a constant (E) on the left-hand side, there must be at least one term on the right-hand side, from which a constant arises. In the most general case this must be one of the terms containing the parameters n, α and β of the Jacobi polynomials. We can make sure that, for example, the first such term gives a constant if $g(x)$ satisfies the differential equation

$$\frac{(g')^2}{1-g^2}=C=\text{constant}. \tag{4.4}$$

We can get three other differential equations from the remaining three terms. Of course, this does not mean that we can find all the possible functions $g(x)$, but this is a convenient way of finding some of them. We can get different kinds of $g(x)$ functions from these differential equations, depending on the sign of C. We list the

Table 1. Potential and energy formulae and wavefunctions deduced using the method described in § 3. We tried to follow the notation of Dabrowska *et al* (1988). ($A = sa$, $B = \lambda a$ for case PI and LIII and $B = \lambda a^2$ for case PII.)

Differential equation (type)	g	$V_{-}(x)$	E_n	$f(x)F(g(x))$
$(g')^2/(1-g^2) = C$	$\mathrm{i}\sinh(ax)$	$A^2 + (B^2 - A^2 - Aa)\,\mathrm{sech}^2(ax)$ $+ B(2A+a)\,\mathrm{sech}(ax)\tanh(ax)$	$A^2 - (A - na)^2$	$(1-g^2)^{-1/2}\exp(-\lambda tg^{-1}(-\mathrm{i}g))P_n^{(-\mathrm{i}\lambda-s-\frac{1}{2},\mathrm{i}\lambda-s-\frac{1}{2})}(g)$
(PI)	$\cosh(ax)$	$A^2 + (A^2 + B^2 + Aa)\,\mathrm{cosech}^2(ax)$ $- B(2A+a)\,\mathrm{cosech}(ax)\coth(ax)$	$A^2 - (A - na)^2$	$(g-1)^{(\lambda-s)/2}(g+1)^{-(\lambda+s)/2}P_n^{(\lambda-s-\frac{1}{2},-\lambda-s-\frac{1}{2})}(g)$
	$\cos(ax)$	$-A^2 + (A^2 + B^2 - Aa)\,\mathrm{cosec}^2(ax)$ $- B(2A-a)\,\mathrm{cosec}(ax)\cot(ax)$	$(A + na)^2 - A^2$	$(1-g)^{(s-\lambda)/2}(1+g)^{(s+\lambda)/2}P_n^{(s-\lambda-\frac{1}{2},s+\lambda-\frac{1}{2})}(g)$
	$\cos(2ax)$	$-(A+B)^2 + A(A-a)\sec^2(ax)$ $+ B(B-a)\,\mathrm{cosec}^2(ax)$	$(A + B + 2na)^2 - (A+B)^2$	$(1-g)^{\lambda/2}(1+g)^{s/2}P_n^{(\lambda-\frac{1}{2},s-\frac{1}{2})}(g)$
	$\cosh(2ax)$	$(A-B)^2 - A(A+a)\,\mathrm{sech}^2(ax)$ $+ B(B-a)\,\mathrm{cosech}^2(ax)$	$(A-B)^2 - (A - B - 2na)^2$	$(g-1)^{\lambda/2}(g+1)^{-s/2}P_n^{(\lambda-\frac{1}{2},-s-\frac{1}{2})}(g)$
$(g')^2/(1-g^2)^2 = C$	$\tanh(ax)$	$A^2 + B^2/A^2 - A(A+a)\,\mathrm{sech}^2(ax)$ $+ 2B\tanh(ax)$, $(\bar{a} = \lambda/(s-n))$	$A^2 + B^2/A^2 - (A+na)^2$ $- B^2/(A+na)^2$	$(1-g)^{(s-n+\bar{a})/2}(1+g)^{(s-n-\bar{a})/2}$ $\cdot P_n^{(-s-n+\bar{a},-s-n-\bar{a})}(g)$
(PII)	$\coth(ax)$	$A^2 + B^2/A^2 + A(A-a)\,\mathrm{cosech}^2(ax)$ $-2B\coth(ax)$, $(\bar{a} = \lambda/(s+n))$	$A^2 + B^2/A^2 + (A+na)^2$ $- B^2/(A+na)^2$	$(g-1)^{-(s+n-\bar{a})/2}(g+1)^{-(s+n+\bar{a})/2}$ $\cdot P_n^{(s-n+\bar{a},-s-n-\bar{a})}(g)$
	$-\mathrm{i}\cot(ax)$	$-A^2 + B^2/A^2 + A(A+a)\,\mathrm{cosec}^2(ax)$ $-2B\cot(ax)$, $(\bar{a} = \lambda/(s-n))$	$-A^2 + B^2/A^2 + (A-na)^2$ $- B^2/(A-na)^2$	$(g^2-1)^{(s-n)/2}\exp(\bar{a}ax)$ $\cdot P_n^{(s-n+\mathrm{i}\bar{a},s-n-\mathrm{i}\bar{a})}(g)$
$(g')^2/g = C$ (LI)	$\frac{1}{2}\omega x^2$	$\frac{1}{4}\omega^2x^2 + l(l+1)/x^2 - (l+\frac{3}{2})\omega$	$2n\omega$	$g^{(l+1)/2}\exp(-g/2)L_n^{(l+\frac{1}{2})}(g)$
$(g')^2 = C$ (LII)	$e^2x/(n+l+1)$	$\frac{1}{4}e^4/(l+1)^2 - e^2/x + l(l+1)/x^2$	$\frac{1}{4}e^2/(l+1)^2 - \frac{1}{4}e^2/(n+l+1)^2$	$g^{l+1}\exp(-g/2)L_n^{(2l+1)}(g)$
$(g')^2/g^2 = C$	$2B/a\exp(-ax)$	$A^2 - B(2A+a)\exp(-ax)$ $+ B^2\exp(-2ax)$	$A^2 - (A - na)^2$	$g^{s-n}\exp(-g/2)L_n^{(2s-2n)}(g)$
$(g')^2 = C$ (HI)	$(\frac{1}{2}\omega)^{1/2}(x - 2b/\omega)$	$-\frac{1}{2}\omega + \frac{1}{4}\omega^2x^2$	ωn	$\exp(-\frac{1}{2}g^2)H_n(g)$
$(g')^2g^2 = C$ (HII)	$2C^{1/2}x^{1/2}$	$4C^2 - C/x - \frac{3}{16}(1/x^2)$	$4C^2 - 4C^2/(2n+1)^2$	$g^{1/2}\exp(-\frac{1}{2}g^2)H_n(g)$

possible $g(x)$ arising from the first two such differential equations in table 1. When solving the third differential equation

$$\frac{(g')^2 g}{(1-g^2)^2}=C$$

it turns out that the function $x(g)$ obtained from it cannot be inverted. Thus it is not possible to obtain a usable $g(x)$ in this case. The fourth differential equation

$$\frac{(g')^2 g^2}{(1-g^2)^2}=C$$

yields

$$g(x)=\{1-\exp[-2(ax+b)]\}^{1/2}$$

and

$$g(x)=\{1-\exp[-2\mathrm{i}(ax+b)]\}^{1/2}$$

where $C=a^2$ and $C=-a^2$, respectively and b is a constant of integration. It can be shown that the latter $g(x)$ gives complex terms when it is substituted into (4.3), while it turns out that the former one has to be turned down for some other reasons mentioned later.

Substituting the $g(x)$ listed in table 1 into (4.3) we can deduce explicit expressions for E_n and $V(\alpha, \beta, x)$. When the constant term in (4.3) is different from the one containing n, we have to shift the n dependence to the constant term and at the same time we have to rid the remaining terms of n. This can be carried out by a transformation of the parameters. These transformations determine the n dependence of the spectrum in each case.

Here we illustrate this procedure with the example of $g(x)=-\mathrm{i}\cot(ax)$ which leads us to a potential missing from the compilation of Dabrowska *et al* (1988).

From (4.3) we get

$$E-V(x)=a^2[(p-\tfrac{1}{2})^2+q^2]+a^2[2\mathrm{i}q(p-\tfrac{1}{2})]\cot(ax) \\ -a^2(n+p+\tfrac{1}{2})(n+p-\tfrac{1}{2})\operatorname{cosec}^2(ax) \qquad (4.5)$$

where $p=\frac{1}{2}(\alpha+\beta+1)$ and $q=\frac{1}{2}(\beta-\alpha)$.

Introducing $s=n+p-\frac{1}{2}$ and $\lambda=\mathrm{i}q(s-n)$ as new parameters we can transfer the n dependence to the constant term E:

$$E-V(x)=a^2\left((n-s)^2-\frac{\lambda^2}{(n-s)^2}\right)-a^2s(s+1)\operatorname{cosec}^2(ax)+2a^2\lambda\cot(ax). \qquad (4.6)$$

Introducing notations similar to those used by Dabrowska *et al* (1988) we obtain the following expressions for E and $V(x)$ (making sure that $E_{n=0}=0$):

$$E=-A^2+\frac{B^2}{A^2}+(A-na)^2-\frac{B^2}{(A-na)^2} \qquad (4.7)$$

and

$$V(x)=-A^2+B^2/A^2+A(A+a)\operatorname{cosec}^2(ax)-2B\cot(ax) \qquad (4.8)$$

where $A=sa$ and $B=\lambda a^2$.

Table 1 contains all the potential and energy formulae obtained in a similar way from the other $g(x)$. It is worth mentioning that terms arising from $(g')^2 g/(1-g^2)^2$ in (4.3) usually 'spoil' the symmetry of the potentials. These terms can be eliminated as taking $\alpha = \beta$, which is an obvious restriction of $P_n^{(\alpha,\beta)}(g(x))$ leading to the less general case of Gegenbauer polynomials.

From (3.10) we get

$$f(x) \simeq (g')^{-1/2}(1+g)^{(\beta+1)/2}(1-g)^{(\alpha+1)/2}. \tag{4.9}$$

Applying the same procedure to the generalised Laguerre polynomials does not reveal anything new. The solutions of differential equations similar to (4.4) do not produce so large a variety of $g(x)$ as in the case of the Jacobi polynomials. We obtain only the three-dimensional Coulomb and oscillator potentials and the Morse potential (with $l=0$), which all have solutions containing confluent hypergeometric functions. This is not surprising, since the generalised Laguerre polynomials $L_n^{(\alpha)}(g)$ are special cases of ${}_1F_1(a, c; g)$ with $a=-n$ and $c=\alpha+1$. In this case $f(x)$ is given by

$$f(x) \simeq (g')^{-1/2} g^{(\alpha+1)/2} \exp(-\tfrac{1}{2}g). \tag{4.10}$$

Taking the Hermite polynomials, similar calculations yield the one-dimensional harmonic oscillator potential and a potential similar to the Kratzer potential for $l=0$. This restriction is in connection with the fact that the Hermite polynomials have one less parameter than the generalised Laguerre polynomials (or the confluent hypergeometric function) which provide us with the general solutions of the Kratzer potential.

Here

$$f(x) \simeq (g')^{-1/2} \exp(-\tfrac{1}{2}g^2). \tag{4.11}$$

The procedure described here can be applied on any special function which satisfies a homogeneous linear second-order differential equation. Applying it on the hypergeometric function ${}_2F_1(a, b; c; g(x))$ we are led to a family of solvable potentials which overlaps with the set of potentials obtained from the Jacobi equation.

Before turning to more general considerations we remark that many of these potentials have been found by Infeld and Hull (1951) in their famous paper on the factorisation method. (This method has been generalised since then in many respects. See, for example, Humi (1987, 1988) and references therein.) In table 2 we also indicate the type of factorisation due to Infeld and Hull. It is interesting to see that each factorisation type corresponds to a differential equation for $g(x)$, similar to (4.4), namely type A and E factorisations can be related to two such differential equations which arose when we applied the method described in § 3 to the Jacobi polynomials; type B, C and F factorisations can be related to three such differential equations arising from a similar treatment of the generalised Laguerre polynomials, and finally type D factorisation can be related to the same treatment of Hermite polynomials.

Miller (1968) also carried out a classification of factorisation types based on the Lie theory of special functions. In his system of classification he relates different factoristion types to different Lie algebras and special functions (hypergeometric, confluent hypergeometric and Bessel functions). In the classification scheme of Miller, type C″ and type D″ factorisation types are related to the Bessel functions. There are no examples for these factorisation types in table 1. Otherwise our results are compatible with this classification scheme, since the Jacobi polynomials can be obtained from the hypergeometric function as special cases and the same holds for the generalised Laguerre and Hermite polynomials and the confluent hypergeometric function. We

Table 2. Comparison of different classification schemes of factorisation types and shape-invariant potentials. In the last two columns the related special functions and Lie algebras (due to Miller 1968) are presented.

$g(x)$	$W(x)$	Infeld and Hull (1951)	Miller (1968)	Cooper *et al* (1987)	Present work	Special function	Lie algebra
$\mathrm{i}\sinh(ax)$	$B\,\mathrm{sech}(ax)+A\tanh(ax)$		A	(b)(iii)	PI	${}_2F_1$	$\mathscr{G}(1,0)\simeq\mathrm{sl}(2)\oplus\mathscr{E}$
$\cosh(ax)$	$-B\,\mathrm{cosech}(ax)+A\coth(ax)$		A	(b)(iv)	PI	${}_2F_1$	$\mathscr{G}(1,0)\simeq\mathrm{sl}(2)\oplus\mathscr{E}$
$\cos(ax)$	$B\,\mathrm{cosec}(ax)-A\cot(ax)$	A	A	(b)(v)	PI	${}_2F_1$	$\mathscr{G}(1,0)\simeq\mathrm{sl}(2)\oplus\mathscr{E}$
$\cos(2ax)$	$A\tan(ax)-B\cot(ax)$	A	A	(d)(ii)	PI	${}_2F_1$	$\mathscr{G}(1,0)\simeq\mathrm{sl}(2)\oplus\mathscr{E}$
$\cosh(2ax)$	$A\tanh(ax)-B\coth(ax)$	A	A	(d)(i)	PI	${}_2F_1$	$\mathscr{G}(1,0)\simeq\mathrm{sl}(2)\oplus\mathscr{E}$
$\tanh(ax)$	$B/A+A\tanh(ax)$	E	E	(c)(i)	PII	${}_2F_1$	$\mathscr{T}_6$
$\coth(ax)$	$B/A-A\coth(ax)$	E	E	(c)(iii)	PII	${}_2F_1$	$\mathscr{T}_6$
$-\mathrm{i}\cot(ax)$	$-B/A+A\cot(ax)$				PII	${}_2F_1$	$\mathscr{T}_6$
$\frac{1}{2}\omega x^2$	$\frac{1}{2}\omega x-\frac{l+1}{x}$	C	C′	(b)(i)	LI	${}_1F_1$	$\mathscr{G}(0,1)$
$\frac{e^2}{n+l+1}$	$-\frac{l+1}{x}+\frac{e^2}{2(l+1)}$	F	F	(c)(ii)	LII	${}_1F_1$	$\mathscr{T}_6$
$\frac{2B}{a}\exp(-ax)$	$A-B\exp(-ax)$	B	B	(b)(ii)	LIII	${}_1F_1$	$\mathscr{G}(1,0)\simeq\mathrm{sl}(2)\oplus\mathscr{E}$
$(\omega/2)^{1/2}(x-2b/a)$	$\frac{1}{2}\omega x-b$	D	D′	(a)	HI	${}_1F_1$	$\mathscr{G}(0,1)$

presented the above classification schemes, together with the classification system of Cooper *et al* (1987) for shape-invariant potentials, in table 2. We also presented the related special functions (and orthogonal polynomials) and Lie algebras due to Miller (1968). It is interesting that the $W(x)=-B/A+A\cot(ax)$ ($g(x)=-\mathrm{i}\cot(ax)$) case has escaped notice, although it is easy to fit it into any of the classification schemes.

It would be interesting to study the relationship between Lie algebras investigated by Miller (1968) and graded Lie algebras which give a mathematical framework of supersymmetry (see, for example, Kac 1977), but this is beyond the scope of this work. The $\mathrm{sl}(1/1)\otimes\mathrm{su}(2)$ algebra studied by Cooper *et al* (1987) in connection with the problem of factorisation seems to be a promising possibility.

As for the superpotential $W(x)$, it is easy to express it in terms of $g(x)$ and $Q(x)$ from (3.10) and (3.12):

$$W(x)=-\tfrac{1}{2}Q(g(x))g'(x)+\tfrac{1}{2}g''(x)/g'(x). \tag{4.12}$$

We used this equation to give explicit expression of $W(x)$ for each case in table 3.

It has already been mentioned that, whenever $R(x)$ vanishes for the ground state, $V(x)$ can be expressed as $V(x)=W^2(x)-W'(x)$. In the case of orthogonal polynomials, this is automatically satisfied for $n=0$. If we take the hypergeometric function ${}_2F_1(a, b; c; g(x))$, this condition is equivalent with $ab=0$ for the ground state. This is the case for most of the potentials with solutions containing the hypergeometric function (or at least it can be achieved after a transformation of the parameters using identity relations of hypergeometric functions). But there are some exceptions, for example in the case of the Woods-Saxon potential (with $l=0$), ab is not 0, so this is an obvious counterexample. There are some other indications that this must be the case. For example, there is no explicit expression for E_n for the Woods-Saxon potential, only a transcendent equation which has to be solved in order to obtain eigenvalues of E (see Flügge 1971).

There are some other potential problems as well with the same character, namely that we know the explicit expression of wavefunctions, but E_n is not known explicitly, for example, in the case of potentials with solutions related to the Bessel functions ($V(x)=\exp(-x/a)$, $l=0$, or a particle enclosed in a sphere). It is easy to see why these potentials are missing from our treatment. For the Bessel functions $R(\nu, x)=1-\nu^2/x^2$, so $R(\nu, x)$ cannot be set to zero, irrespective of the choice of parameter ν. For this reason it is not possible to cast $V(x)$ in the form $V(x)=W^2(x)-W'(x)$.

5. Search for shape-invariant potentials

It is easy to prove within the framework of the simple method presented in § 3 that the shape-invariance condition (2.8) can be satisfied by almost every potential type obtained here. To this end we express the superpotential $W(x)$ in terms of $g(x)$. This can be carried out combining equation (4.12) and equations similar to (4.4) which enable us to express $g'(x)$ as a function of $g(x)$. Here one has to be cautious since, when expressing g', an ambiguity of signs arises so the signs have to be determined separately for each case. We listed the g' and $W(g)$ as functions of g in table 3. In the next step one can calculate $V_+(\{a_0\}, g)-V_-(\{a_1\}, g)$ with the constraint that this must give a constant. This means that the g dependence must vanish from this expression, and this condition provides us with a relationship between the sets of parameters $\{a_1\}$ and $\{a_0\}$ in each case. One also has to apply the appropriate change

Table 3. Determination of $R(a_k)$ and $\{a_k\}$, in each case applying the shape-invariance condition (2.8) (see text for details).

$F(g)$ (type)	$W(\{a_k\}, g)$	$V_\pm(\{a_k\}, g)$	g	g'	C	$\{a_k\}$	$R(a_k)$
$P_n^{(\alpha,\beta)}(g)$	$\dfrac{C^{1/2}}{(1-g^2)^{1/2}}(-q+pg)$	$\dfrac{C}{1-g^2}[(q^2+p^2\pm p)-q(2p\pm 1)g$ $-p^2(1-g^2)]$	$\mathrm{i}\sinh(ax)$	$C^{1/2}(1-g^2)^{1/2}$	$-a^2$	$p_k=-(s_0-k)$, $q_k=q_0$	$C[p_k^2-(p_k-1)^2]$
(PI)			$\cosh(ax)$	$-C^{1/2}(1-g^2)^{1/2}$	$-a^2$	$p_k=-(s_0-k)$, $q_k=q_0$	
			$\cos(ax)$	$-C^{1/2}(1-g^2)^{1/2}$	a^2	$p_k=(s_0+k)$, $q_k=q_0$	
			$\cos(2ax)$	$-C^{1/2}(1-g^2)^{1/2}$	$4a^2$	$p_k=\frac{1}{2}(s_0+\lambda)+k$, $q_k=q_0$	
			$\cosh(2ax)$	$-C^{1/2}(1-g^2)^{1/2}$	$-4a^2$	$p_k=\frac{1}{2}(\lambda-s_0)+k$, $q_k=q_0$	
(PII)	$C^{1/2}[-q+(p-\frac{1}{2})g)]$	$C\{[q^2+(p-\frac{1}{2})^2]-q(2p-1)g$ $-(p-\frac{1}{2})(p-\frac{1}{2}\mp 1)(1-g^2)\}$	$\tanh(ax)$	$C^{1/2}(1-g^2)$	a^2	$p_k=s_0-k+\frac{1}{2}$, $\Lambda=-\lambda$	$C((p_k+\frac{1}{2})^2-(p_k-\frac{1}{2})^2$ $+\Lambda^2/(p_k+\frac{1}{2})^2$ $-\Lambda^2/(p_k-\frac{1}{2})^2$
			$\coth(ax)$	$C^{1/2}(1-g^2)$	a^2	$p_k=-s_0-k+\frac{1}{2}$, $\Lambda=-\lambda$	
		$(q_k(p_k-\frac{1}{2})=\Lambda)$	$-\mathrm{i}\cot(ax)$	$C^{1/2}(1-g^2)$	$-a^2$	$p_k=s_0-k+\frac{1}{2}$, $\Lambda=-\mathrm{i}\lambda$	
$L_n^{(\alpha)}(g)$ (LI)	$C^{1/2}[-\frac{1}{4}(2\alpha+1)g^{-1/2}+\frac{1}{2}g^{1/2}]$	$C\{-\frac{1}{4}(2\alpha+1)\pm\frac{1}{4}+[\frac{1}{4}(2\alpha+1)]$ $\cdot[\frac{1}{4}(2\alpha+1)\pm\frac{1}{2}]g^{-1}+\frac{1}{4}g\}$	$\frac{1}{2}Cx^2$	$C^{1/2}g^{1/2}$	2ω	$\alpha_k=l_0+k+\frac{1}{2}$	$C=2\omega$
(LII)	$C^{1/2}(\alpha)[-\frac{1}{2}(\alpha+1)g^{-1}(\alpha)+\frac{1}{2}]$	$C(\alpha)\{\frac{1}{4}-\frac{1}{2}(\alpha+1)g^{-1}(\alpha)+[\frac{1}{2}(\alpha+1)]$ $\cdot[\frac{1}{2}(\alpha+1)\pm 1]g^{-2}(\alpha)\}$	$C^{1/2}(\alpha)x$	$C^{1/2}(\alpha)$	$(2e^2/(\alpha+1))^2$	$\alpha_k=2(l_0+k)+1$	$[C(\alpha_k-2)-C(\alpha_k)]/4$
(LIII)	$C^{1/2}(\frac{1}{2}\alpha-\frac{1}{2}g)$	$C(\frac{1}{4}\alpha^2-[\frac{1}{2}(\alpha\mp 1)]g+\frac{1}{4}g^2)$	$\exp(-C^{1/2}x+B)$	$-C^{1/2}g$	a^2	$\alpha_k=2(s_0-k)-2n$	$C[(\alpha_k+2)^2-\alpha_k^2]/4$
$H_n(g)$							
(HI)	$C^{1/2}g$	$C(g^2\pm 1)$	$C^{1/2}x-B$	$C^{1/2}$	$\omega/2$		$2C=\omega$
(HII)	$C^{1/2}(1-\frac{1}{2}g^{-2})$	$C[1-g^{-2}+(\frac{1}{4}\pm 1)g^{-4}]$	$[2(C^{1/2}x+B)]^{1/2}$	$C^{1/2}g^{-1}$			$2Cg^{-4}\neq$ constant

of parameters mentioned in § 4. Another important thing is that, in some cases, $g(x)$ itself depends on the parameters, for example in the case of the three-dimensional Coulomb problem (see tables 1 and 3).

We illustrate this procedure of obtaining $W(g)$ with the family of potentials arising from $g(x)$ obtained from the differential equations $(g')^2 = C(1-g^2)$ in the case of the Jacobi polynomials ($g(x) = \tanh(ax)$, $\coth(ax)$ and $-\mathrm{i}\cot(ax)$). After expressing $V_+(p, q, g)$ and $V_-(p, q, g)$ (see table 3) we recall that $q(p-\frac{1}{2}) = \Lambda$ is constant (see, for example, (4.5) and (4.6)) ($\Lambda = -\lambda$, λ and $-\mathrm{i}\lambda$ in the case of $g(x) = \tanh(ax)$, $\coth(ax)$ and $-\mathrm{i}\cot(ax)$, respectively). Now, imposing the shape-invariance condition (2.8) we get

$$\begin{aligned} V_+(p_0, q_0, g) - V_-(p_1, q_1, g) &= C[(p_0-\tfrac{1}{2})^2 + q_0^2 - (p_1-\tfrac{1}{2})^2 - q_1^2] + C(-2\Lambda_0 + 2\Lambda_1)g \\ &\quad + C[(p_1-\tfrac{1}{2})(p_1+\tfrac{1}{2}) - (p_0-\tfrac{3}{2})(p_0-\tfrac{1}{2})](1-g^2) \\ &= R(p_1, q_1) = \text{constant}. \end{aligned} \tag{5.1}$$

When we try to eliminate the g dependence from $R(p_1, q_1)$ we are led to the following conditions:

$$q_0(p_0-\tfrac{1}{2}) = \Lambda_0 = \Lambda_1 = q_1(p_1-\tfrac{1}{2}) \tag{5.2}$$

and

$$(p_1-\tfrac{1}{2})(p_1+\tfrac{1}{2}) = (p_0-\tfrac{3}{2})(p_0-\tfrac{1}{2}). \tag{5.3}$$

From (5.3) we get either $p_1 = p_0 - 1$ or $p_1 = -1 - p_0$, but only the former possibility is reasonable from our point of view. From (5.2)

$$q_0 = \frac{\Lambda_0}{p_0-\frac{1}{2}} = \frac{\Lambda}{p_0-\frac{1}{2}}$$

and

$$q_1 = \frac{\Lambda_1}{p_1-\frac{1}{2}} = \frac{\Lambda}{p_1-\frac{1}{2}}$$

follows. Substituting these into (5.2) we get the following expression:

$$R(p_1, \Lambda) = C\left((p_1+\tfrac{1}{2})^2 - (p_1-\tfrac{1}{2})^2 + \frac{\Lambda^2}{(p_1+\frac{1}{2})^2} - \frac{\Lambda^2}{(p_1-\frac{1}{2})}\right). \tag{5.4}$$

We know the exact values of C, Λ and p_1 in each case from table 1, for example when we take $g(x) = -\mathrm{i}\cot(ax)$, $C = -a^2$, $\Lambda = -\mathrm{i}\lambda$ and $p_k = s - k + \frac{1}{2}$.

This procedure provides us with $R(a_k)$ and the functional $f(a_k)$ in most cases (see § 2 and table 3). The only exception is $g(x) = [2(C^{1/2}x + B)]^{1/2}$ arising from $(g')^2 g^2 = C$ at the Hermite polynomials. In this case the above procedure fails simply because the number of free parameters is less than what would be needed to eliminate the coordinate dependence from $R(a_k)$. This is not a problem when we apply this procedure on $g(x) = C^{1/2}x - B$, arising from $(g')^2 = C$, since $V_+(g) - V_-(g) = 2C$ is automatically constant in this case.

Here we arrived at the question of the number of parameters. Parameters originate from different places. The most obvious examples are the arguments of the different orthogonal polynomials. Their number is 2, 1 and 0 for $P_n^{(\alpha,\beta)}(g)$, $L_n^{(\alpha)}(g)$ and $H_n(g)$

if we disregard n as a parameter. Another parameter is C which arises from differential equations of the type (3.4) and which gives the length scale of the potentials. In some cases the constant of integration (introduced when solving these differential equations) also plays an important role (for example, $g(x) = (2B/a)\exp(-ax) = \exp(-ax+b)$ at the Morse potential), but in most cases it is only a coordinate shift. Therefore the number of significant parameters is 3, 2 or 1.

6. Conclusions

Here we investigated a simple method of finding solvable potentials from the point of view of supersymmetric quantum mechanics. This method can be used to transform the Schrödinger equation into a linear homogeneous second-order differential equation with known special functions as solutions. Combining this method with the theory of supersymmetric quantum mechanics we obtained conditions which must be satisfied by the special functions in order to lead to potentials of the form $V(x) = W^2(x) \pm W'(x)$. These conditions are automatically fulfilled by the orthogonal polynomials. They are also fulfilled by ${}_2F_1(a, b; c; g)$ if $ab = 0$ holds for the ground state. (Potentials obtained from ${}_2F_1(a, b; c; g)$ are, of course, not independent from those obtained from the Jacobi polynomials, since these are special cases of the hypergeometric function.) We could get similar criteria for any special function (satisfying a linear homogeneous second-order differential equation) with an arbitrary number of parameters. This simple method can help us to investigate potential problems with solutions $\Psi(x) = f(x)F(g(x))$. A straightforward generalisation would be to search for $\Psi(x)$ as a linear combination of special functions.

We have shown that applying the above method on orthogonal polynomials we can find examples of every single factorisation type introduced by Infeld and Hull (1951). It would be an interesting task to investigate the relationship of graded Lie algebras and Lie algebras used by Miller (1968) to classify the factorisation types in terms of the Lie theory of special functions.

We have also studied the question of shape invariance in terms of the method described above. It turned out that the shape-invariance condition (2.8) can be satisfied almost in each case, depending on the number of free parameters.

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